**CME 323: Distributed Algorithms and Optimization, Spring 2015**
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**Lecture 18, 5/25/2016. Scribed by Vishakh Hegde, Alex Williams and Patrick Landreman.**

Lecture outline:

- Matrix multiplication on a cluster
- Singular Value Decomposition

# 18 Matrix Multiplication on a Cluster

In the first half of this course, we have seen algorithms to perform matrix-matrix multiplication, assuming the PRAM model. In this lecture we will see how we can perform matrix-matrix multiplication in a distributed manner. In Spark, matrices are typically stored broken up for storage in three different ways:

- By entries (CoordinateMatrix): stored as a list of (i, j, value) tuples
- By rows (RowMatrix): each row is stored separately (e.g. Pagerank)
- By blocks (BlockMatrix): by storing submatrices of a matrix as dense matrices, block matrices can take advantage of low-level linear algebra library for operations like multiplications.

Consider two matrices A, of size $m \times k$ and B, of size $k \times n$. The matrix product $AB$ will have size $m \times n$. Let these matrices be saved as an RDD of $(i, j, value)$ corresponding to the $ij^{th}$ element of $A$ and $(p, q, value)$ corresponding to the $pq^{th}$ element of $B$. The $iq^{th}$ element of the product $AB$ is $(AB)_{iq} = \sum_{i=1}^{k} A_{ij}B_{jq}$. So it is clear that in order to perform a reduce operation (summation) in the MapReduce framework, we first need to get these values to have the same key. Therefore, we need to produce key-value pairs with the value being $A_{ij}B_{jq}$ and the key being $(i, q)$ for all possible values of $j$. The Map phase will constitute an all to all communication. We then do a ReduceByKey operation to sum up all values corresponding to the same key.

# 19 Singular Value Decomposition (SVD)

The rank-$r$ singular value decomposition (SVD) is a factorization of a real matrix $A \in \mathbf{R}^{m \times n}$, such that $A = U\Sigma V^T$, where $U \in \mathbf{R}^{m \times r}$ and $V \in \mathbf{R}^{n \times r}$ are unitary matrices holding the left and right singular vectors of $A$, and $\Sigma$ is a $r \times r$ diagonal matrix with non-negative real entries, holding the top $r$ singular values of $A$.

### 19.1 When $A$ is a `RowMatrix`

It is easy to see that the singular values and right singular vectors can be recovered from the SVD of the Gramian matrix $A^TA$:

$$A^TA = (U\Sigma V^T)^T U\Sigma V^T = V\Sigma U^T U\Sigma V^T = V\Sigma^2 V^T$$

We can exploit this property to efficiently compute the SVD of a tall-skinny matrix. If $n$ is small enough to fit on a single machine, then $A$ can be distributed as a one-dimensional block-row matrix (in Spark this is called a `RowMatrix`). We compute $A^TA$ using the methods described in the previous section, then solve the SVD of $A^TA$ locally on a single machine to determine $\Sigma$ and $V$. Finally, we can solve for $U$ by simple matrix multiplications:

$$A = U\Sigma V^T \;\Rightarrow\; U = AV\Sigma^{-1}$$

The key observation is that $A^TA$ is a much smaller matrix than $A$ when $n \ll m$. In general, computing the rank-$r$ SVD of $A$ will cost $O(mnr)$ operations (not to mention expensive communication costs), while this computation only costs $O(n^2r)$ operations for $A^TA$. Of course, if $A$ is short and fat (i.e. $n \gg m$), we can run this same algorithm on $A^T$. The complete algorithm is given below:

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**Procedure 1** SVD of a `RowMatrix`

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**Input:** $A \in \mathbf{R}^{m\times n}$ and desired rank $r$. $A$ is distributed as a `RowMatrix` across $p$ machines, machine $\ell$ holds a submatrix $A^{(\ell)} \in \mathbf{R}^{\frac{m}{p}\times n}$.

**Output:** The singular value decomposition of $A$. $U$ is distributed as a `RowMatrix` similar to $A$.

1: Each machine computes $A^{(\ell)T}A^{(\ell)}$

2: ***reduce:*** $A^TA = \sum_{\ell=1}^{p} A^{(\ell)T}A^{(\ell)}$

▷ The result is a small $n \times n$ matrix

3: $V\Sigma^2V^T = A^TA$

4: $\Sigma = \sqrt{\Sigma^2}$

5: ***broadcast:*** $V\Sigma^{-1}$

6: Each machine computes $U^{(\ell)} = A^{(\ell)}V\Sigma^{-1}$

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*Communication costs:* We assume that we use combiners so that each machine locally computes its portion of $A^TA$ in line 1 separately before communication between machines occurs in line 2. The only communication costs are the all-to-one communication on line 2 with message size $n^2$, and the one-to-all communication on line 5 with message size $nr$. These require $O(\log p)$ messages when a recursive doubling communication pattern is used.

### 19.2 When $A$ is square

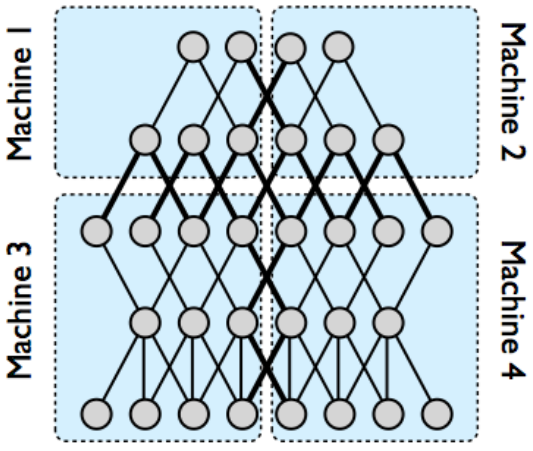
The algorithm which produces the $k$ largest eigenvectors of $A^TA$ is implemented with ARPACK and requires successive calls of matrix-vector multiplication. In the tall & skinny case above, $A^TA$ is relatively small and can be stored locally on the Driver machine. In the case where $A$ is square or nearly square, however, $A^TA$ remains a large matrix and the rows must be distributed. In this situation, the updated vector $(A^TA)v$ is produced by the following:

- Broadcast v to all workers
- Each worker computes the $i$th element of $Av = a_i^T v$ according to whichever rows of $A$ are present
- The $i$th column of $A^T$ is then scaled by the corresponding result of the previous dot product. Since the columns of $A^T$ are simply the rows of $A$, no additional data is needed
- The scaled columns are reduced by summation on the Driver

The above process is implemented using an `Aggregate` method. This distributed solution requires passing data over the network $O(k)$ times, compared to $O(1)$ when the Gramian matrix is stored locally. In exchange, the storage requirement is reduced from $O(n^2)$ to $O(n)$.

## 20 Distributed Neural Networks

Neural networks (NN) are an extremely active subject of research, enabled by advancements in data storage and parallel computing power. NNs may be distributed via two schemes: data parallelism or model parallelism. In the former, the nodes of the NN itself can reside on a single computer, while the training and input data (possibly large image files, etc) are scattered across a distributed file system. In model parallelism, the NN itself is too large to contain on a single computer. In this case, the model is clustered into blocks which communicate between each other using vectors - each element in a vector represents one input or output *"synapse"* of the block. Below is an example from DistBelief, a distributed NN developed by Google, inc. The figure illustrates a set of nodes which have been divided amongst four machines. The links in bold highlight synapses which communicate via input and output vectors.



## References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